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A REFINED MODEL FOR BEAMS ON ELASTIC FOUNDATIONS

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Abstract—The concept of a beam or slab on an elastic foundation has been one of the convenient tools for obtaining solutions to several geotechnical engineering problems. While it is easy to establish quite accurately the stiffness characteristics of the beam or the slab, the parameters which govern the behavior of the subsoil or the elastic foundation are indeed hard to model. This difficulty remains true even if one assumes that the soil has linear, isotropic, and homogeneous properties. Using the Vlasov model, Vallabhan and Das (1988, *J. Engng Mech. Div., ASCE* 114(2), 2072–2082) have developed an effective iterative technique to solve the beam-on-elastic-foundation problem where the soil is assumed to have a uniform depth with a rigid base at the bottom. Geotechnical engineers usually encounter subsoil of finite depth, and normally the elastic properties are considered to remain constant or vary linearly with depth. In this paper, the authors have extended their model to incorporate numerically this physical characteristic of the soil. Even though the derivations may look laborious, the numerical model is quite simple and can be programmed on a desktop computer.

INTRODUCTION

Many geotechnical engineers use the Winkler model (1867) for the analysis of beams on elastic foundations, where the vertical deformation characteristics of the elastic foundation are characterized by means of continuous, closely spaced linear springs; the constant of proportionality of these springs is known as the "modulus of subgrade reaction" (Terzaghi, 1955), and is usually represented by k. The Winkler model has been shown to be inconsistent to represent a continuous medium (see Pasternak, 1954; Vlasov and Leont'ev, 1966; and recently Nogami and Lam, 1987). Vlasov developed a two-parameter model in which he introduced an arbitrary parameter, γ, to characterize the distribution of displacements in the vertical direction in the elastic foundation. Jones and Xenophontos (1977), using the variational principle, strengthened the Vlasov model by establishing a relationship between the γ-parameter and the displacement of the beam or the slab on the top. Vallabhan and Das (1988) developed an iterative procedure to uniquely determine the γ-parameter. The two Vlasov parameters and the γ-parameter are interdependent of one another and unique for a given problem of beam or slab on elastic foundation.

In this paper, the authors consider an elastic foundation of finite depth on a rigid base and assume the elastic properties of the layer to vary linearly with depth. Here, one should be reminded that the elastic foundation is assumed to be a plane strain problem; in other words, mathematically the model is for solving a plane strain problem of a slab on a finite layer of elastic continuum with linearly varying elastic modulus.

WHY A NEW MODEL?

In spite of its conceptual elegance, the Vlasov model has not been used by many practicing engineers, even though it was developed as early as 1966. The authors presume the following four major reasons to account for the lack of acceptance of the Vlasov model in engineering design. The first reason relates to the difficulties in the estimation of the value of the γ-parameter which defines the decay of the vertical displacement in the subsoil. The second reason is that the exact solutions of the Vlasov equations are rather complex and difficult to achieve; a proper numerical procedure has to be employed. The third reason is that the soil properties need not be uniform for the entire depth; in many instances, they vary with depth. The fourth reason, according to the authors, is the non-availability of the comparison of the solution of the Vlasov model with other more exact solutions. Vallabhan *et al.* (1987), using the finite difference technique combined with an iterative procedure, removed the above limitations. Also, Vallabhan and Das (1987) extended this concept to a "matrix foundation model" whereby the accuracy of their model was further improved. Some of the observations from fellow researchers on these models were attributed to the practicality of these concepts to solve real geotechnical problems. One of the primary suggestions was directed to the nature of the soil layers and their variations in elastic properties with depth. According to the authors, such realistic situations can be approximated, in many instances, by assuming linear variations of elastic properties with depth. The incorporation of this linear variation into the three models is the theme of this paper.

VLASOV MODEL

In order to recapitulate, a brief account of the Vlasov model is given below. The subsoil is assumed to have uniform depth with linearly varying elastic modulus resting on a stiff soil layer or bedrock. The derivations are made for the case of a long slab of finite width resting on the elastic foundation, thus assuming plane strain conditions for the soil as shown in Fig. 1. A strip of the slab (of width b in the y-direction) is assumed as a beam. The potential energy function is given by

$$\Pi = \int_0^L \frac{E_b I_b}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2 dx + \frac{b}{2}\int_{-\infty}^{\infty}\int_0^H (\sigma_x\varepsilon_x + \sigma_z\varepsilon_z + \gamma_{xz}\tau_{xz})\, dz\, dx - \int_0^L q(x)\bar{w}(x)\, dx \quad (1)$$

where $E_b I_b$ = flexural stiffness of the beam, L = length of the beam, H, b = height and width of the soil model, σ_x, σ_z, τ_{xz} = components of stress at a point in the soil, and ε_x, ε_z, γ_{xz} = corresponding strains in the soil. Assumptions:

1. The vertical displacement $\bar{w}(x, z)$ is expressed as $\bar{w}(x, z) = w(x) \cdot \phi(z)$, such that $\phi(0) = 1$ and $\phi(H) = 0$.
2. The horizontal displacement $u(x, z)$ is assumed to be zero everywhere in the soil medium.
3. Compatibility of vertical displacements at the soil beam interface alone is assumed to be of importance.

Minimizing the function Π with respect to w, we get, for the beam $0 < x < L$, the following equation:

$$\frac{d^2}{dx^2}\left(E_b I_b \frac{d^2\bar{w}}{dx^2}\right) - 2t\frac{d^2\bar{w}}{dx^2} + k\bar{w} = q(x) \quad (2)$$

with the usual boundary conditions for the conventional Euler beam. In the above equation,

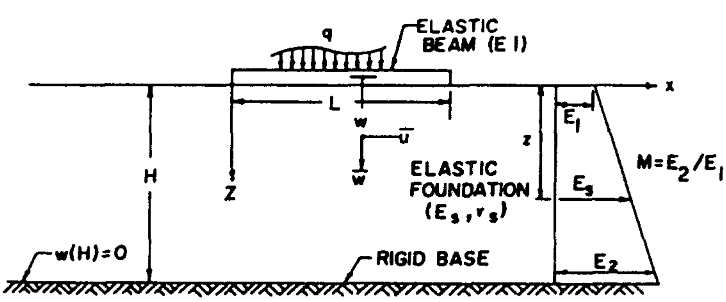

Fig. 1. Beam on elastic foundation (Vlasov model).

$$k = \int_0^H \frac{E_s b(1-\nu)}{(1+\nu)(1-2\nu)} \left(\frac{d\phi}{dz}\right)^2 dz \tag{3}$$

and

$$2t = \int_0^H \frac{E_s b}{2(1+\nu)} \phi^2 \, dz \tag{4}$$

where

$$E_s = E_1\left(1-\frac{z}{H}\right)+E_2\left(\frac{z}{H}\right).$$

Here E_1 and E_2 represent the modulus of elasticity of the top and bottom of the soil stratum respectively. It can be shown (Vallabhan and Das, 1987) that in this model the boundary shear forces are:

$$Q(0) = -\sqrt{2kt}\,\bar{w}(0)$$

and

$$Q(L) = -\sqrt{2kt}\,\bar{w}(L) \quad \text{respectively.} \tag{5}$$

Minimizing the function Π with respect to ϕ and with further algebraic simplifications, it can be shown (Vallabhan and Das, 1988) that

$$\phi(z) = \frac{\sinh \gamma\left(1-\frac{z}{H}\right)}{\sinh \gamma} \tag{6}$$

where

$$\left(\frac{\gamma}{H}\right)^2 = \frac{1-2\nu}{2(1-\nu)} \frac{\int_{-\infty}^{\infty} \left(\frac{d\bar{w}}{dx}\right)^2 dx}{\int_{-\infty}^{\infty} \bar{w}^2 \, dx}. \tag{7}$$

It is to be noted that γ herein is a dimensionless parameter, whereas Vlasov used a dimension of length^{-1}. Using the expression for $\phi(z)$, one can derive new expressions for k and $2t$, namely

$$k = \frac{b(1-\nu)}{8H(1+\nu)(1-2\nu)}\left[\frac{E_1(2\gamma \sinh 2\gamma+4\gamma^2)+(E_2-E_1)(\cosh 2\gamma-1+2\gamma^2)}{\sinh^2 \gamma}\right] \tag{8}$$

$$2t = \frac{bH}{16\gamma^2(1+\nu)}\left[\frac{E_1(2\gamma \sinh 2\gamma-4\gamma^2)+(E_2-E_1)(\cosh 2\gamma-1-2\gamma^2)}{\sinh^2 \gamma}\right]. \tag{9}$$

DETERMINATION OF γ PARAMETER

Expanding eqn (7), it is shown that

$$\left(\frac{\gamma}{H}\right)^2 = \frac{(1-2\nu)}{2(1-\nu)} \frac{\int_0^L \left(\frac{d\bar{w}}{dx}\right)^2 dx + \frac{1}{2}\sqrt{\frac{k}{2t}}[\bar{w}^2(0)+\bar{w}^2(L)]}{\int_0^L \bar{w}^2(x)\,dx + \frac{1}{2}\sqrt{\frac{2t}{k}}[\bar{w}^2(0)+\bar{w}^2(L)]}. \tag{10}$$

Assuming an approximate value of γ initially, the values of k and $2t$ are evaluated using eqns (8) and (9). From the solution of the deflection of the beam, a new value of γ is computed using eqn (10). This new value of γ is again used to compute new values of k and $2t$. The procedure is repeated until two successive values of γ are approximately equal.

FINITE DIFFERENCE MODEL

The classical finite difference method is employed to solve the differential equation given in eqn (2). Matlock and Reese (1960) used a similar approach to obtain solutions for laterally loaded pile foundation problems. By using the central difference operator, for an interior node i within the domain, the following equation is obtained:

$$w_{i-2}-(4+\alpha)w_{i-1}+(6+2\alpha+\beta)w_i-(4+\alpha)w_{i+1}+w_{i+2} = \frac{q_i h^4}{E_b I_b}$$

where

$$\alpha = \frac{2th^2}{E_b I_b}, \quad \beta = \frac{kh^4}{E_b I_b} \tag{11}$$

in which $h = L/N$, and N is the number of segments used in the finite difference model. Here the node i has to be within two nodes of the end nodes. For the nodes close to the boundary on either end, the equations need modifications to contain the boundary conditions at the ends. For convenience, the equations are derived for the right-hand side end of the beam. By incorporating the boundary conditions into the field equation, the finite difference equations at stations $n-1$ and n are obtained as given below:

at node $n-1$,

$$w_{n-3}-(4+\alpha)w_{n-2}+(5+2\alpha+\beta)w_{n-1}-(2+\alpha)w_n = \frac{q_{n-1}h^4}{E_b I_b}+\frac{\bar{M}_n h^2}{E_b I_b} \tag{12}$$

and at node n,

$$w_{n-2}-(2+\alpha)w_{n-1}+\left(1+\alpha+\frac{\beta}{2}+\sqrt{2kt}\cdot\frac{h^3}{EI}\right)w_n = \frac{q_n h^4}{2E_b I_b}-\frac{\bar{M}_n h^2}{E_b I_b}+\frac{\bar{Q}_n h^3}{E_b I_b}. \tag{13}$$

$\bar{M}_n$ and $\bar{Q}_n$ are the prescribed values of the bending moments and shear forces at the end of the beam. Now similar equations can be derived for the end at the left-hand side also. Since the final algebraic equation can be represented by an equidiagonal matrix equation, a recursive method is employed to obtain the solution.

A very simple computer program has been developed. The input data consist of beam properties such as dimensions and E_b, and the soil properties such as E_1, E_2, ν, and H respectively. The program internally calculates soil parameters γ, k, and $2t$, using an iterative technique.

NUMERICAL EXAMPLES

Since the authors (Vallabhan and Das, 1987, 1988) have compared their solutions with more sophisticated numerical solutions and have indicated good correlations, similar exercises are not attempted here. Alternatively, two sets of problems of practical significance are solved. In the first one, the beam carries a uniformly distributed load; in the second, the beam has a concentrated load at the center. These problems have been solved for various values of the depth of the soil and a parameter M ($= E_2/E_1$) which identifies the ratio of the moduli at the bottom to that at the top of the soil layer. The dimensions of the beam and the loading characteristics are kept constant in order to illustrate some meaningful comparisons. The main purpose of these exercises is to demonstrate the versatility of the technique to solve beam-on-elastic-foundation problems, without having the need to establish a value of k. Besides, in the authors' opinion, there is no such deterministic soil property as k. No doubt the profession continues to use it; but it needs to be emphasized that it is erroneous to do so. The dimensions and properties of the beam and the soil are assumed as follows: length of the beam (L) = 100 ft (30.48 m), width of the beam (b) = 1 ft (0.3048 m), depth of the beam (d) = 3 ft (0.9144 m), modulus of elasticity of the beam (E_b) = 432,000 ksf (20.68 GPa), modulus of elasticity of the soil at the top (E_1 = 500 ksf (23.94 MPa), and Poisson's ratio of the soil layer (ν) = 0.2.

The above properties remain the same for all problems solved in this paper. The values of the depth of the soil layer, H, have been taken as 25, 50, and 100 ft, and the values of M are varied as 1, 2, 5, 10, and 20, for each value of depth.

Uniformly distributed load case

The uniformly distributed load q is assumed to be equal to 2 K ft^{-1} (2.71 kN m^{-1}). There are 15 problems; each of them is solved independently. The non-dimensional parameters given by the ratio H/L and $M = (E_2/E_1)$ are the most significant parameters which control the solution here. Other parameters defining the stiffness ratio of the beam and the soil, etc., are also known to be major parameters which control the overall solution; however, here, it is intended to show the effects on the solution due to changes in soil geometry and properties. For convenience of the reader, the values obtained for γ, k, and $2t$ are shown in Table 1 for various H/L values of M ratios. Since the displacements and their patterns are of importance to the analyst, they are also illustrated in Figs 2, 3, and 4 respectively. Here the values of displacements are multiplied by a factor $(1+M)/2$. In other words, the actual displacements are to be divided by $(1+M)/2$, which means that for values of $M > 1$, the displacements are less.

Table 1. Values of γ, k, and $2t$ parameters (for a uniformly distributed load = 2 K/f; L = 100 ft, $E_2/E_1 = M$, E_1 = 500 ksf, E_bI_b = 972,000 kft^2)

Soil depth H (ft)	H/L ratio	$M = E_2/E_1$	γ-parameter	k (ksf)	$2t$ (K)
25	0.25	1	0.377	22.2	1704
		2	0.390	33.1	2122
		5	0.407	65.5	3379
		10	0.417	119.4	6471
		20	0.424	227.3	9655
50	0.50	1	0.578	11.1	3325
		2	0.597	16.4	4125
		5	0.622	32.0	6522
		10	0.637	58.0	10511
		20	0.650	109.9	18476
100	1.00	1	0.883	5.6	6296
		2	0.919	8.1	7726
		5	0.966	15.3	11999
		10	0.993	27.2	19111
		20	1.012	51.0	33311

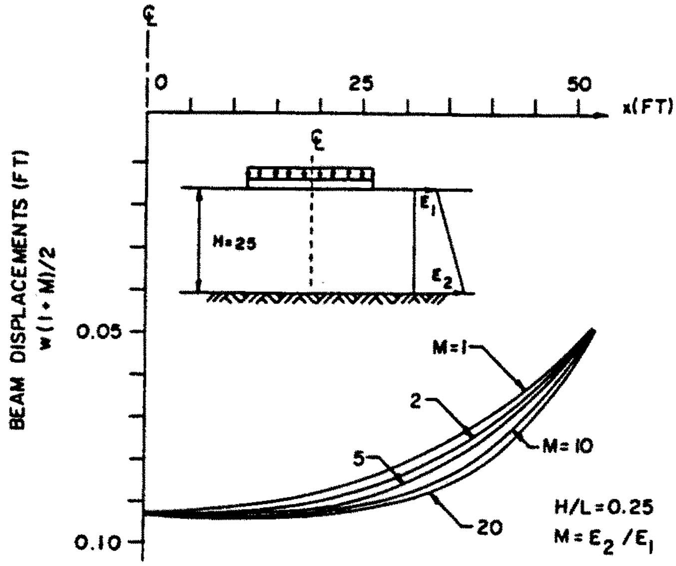

Fig. 2. Beam displacement ($H/L = 0.25$).

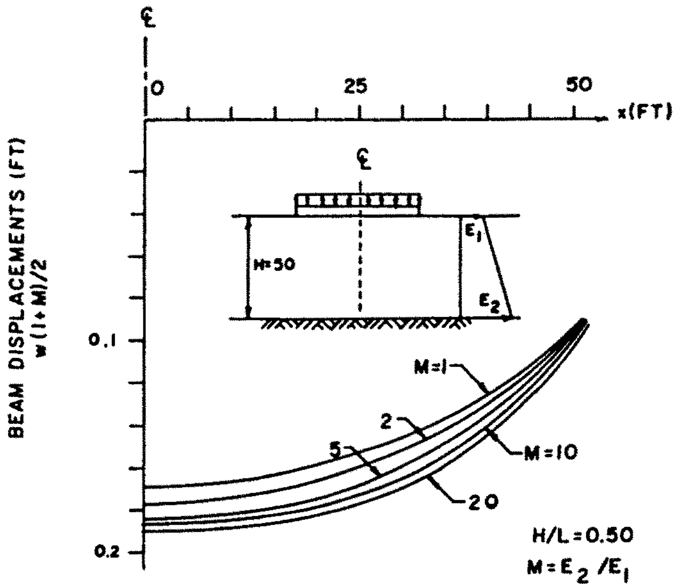

Fig. 3. Beam displacement ($H/L = 0.5$).

Concentrated load case

The second case deals with a concentrated load, $P = 30$ K (133.44 kN), applied at the center of the beam. As in the previous case, there are 15 problems here also. The results containing the variations in the values of γ, k, and $2t$ for different H/L and M ratios are shown in Table 2. Also, the variations of displacements for different H/L ratios are given in Figs 5, 6, and 7 respectively. As before, the values of displacements are multiplied by a factor $(1+M)/2$.

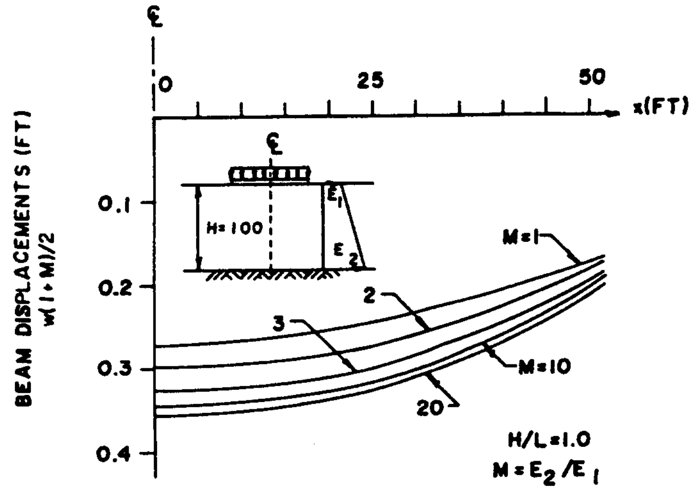

Fig. 4. Beam displacement (H/L = 1.0).

DISCUSSION

The variability of three soil parameters such as γ, k, and $2t$ for different H/L and M ratios is illustrated in Tables 1 and 2. The overall attempt here is to illustrate that beam-on-elastic-foundation problems, where the material properties of the soil layer varies with depth, can be solved numerically by a simple procedure. The conventional technique of establishing a unique numerical value of k for the soil has to be examined carefully, because the value of k varies depending on the overall geometry of the problem and the stiffness ratios of the beam and the soil, etc. Also, for the Winkler model, the displacements are uniform for the case of a uniformly distributed load; in other words, there is no bending moment or shear force in the beam. This phenomenon is contrary to what one gets from using other models.

Many engineers hold the view that the establishment of values for E_1, E_2 and E for the soil stratum is as complex as establishing a value of k for the soil. The authors take a different opinion that the parameter E is conceptually good and that the values can be effectively measured for a soil stratum by means of *in situ* pressure-meter tests or laboratory unconfined and triaxial compression tests. In the case of normally consolidated claying soils, the values of elastic modulus can be less meaningful from a practical point of view, and in such instances a drained modulus may be used so as to obtain time-dependent

Table 2. Values of γ, k, and $2t$ parameters (for concentrated load at the center = 30 K; L = 100 ft, $E_2/E_1 = M$, E_1 = 500 ksf, $E_b I_b$ = 972,000 kft²)

Soil depth H (ft)	H/L ratio	$M = E_2/E_1$	γ-parameter	k (ksf)	$2t$ (K)
25	0.25	1	0.610	22.3	1654
		2	0.657	32.7	2041
		5	0.755	63.0	3169
		10	0.862	111.7	4969
		20	0.992	205.0	83
50	0.50	1	0.94	11.2	3109
		2	1.014	16.1	3773
		5	1.150	29.8	5675
		10	1.285	51.4	8636
		20	1.444	92.1	14100
100	1.00	1	1.345	5.9	5615
		2	1.447	8.0	6652
		5	1.623	14.1	9578
		10	1.781	23.6	14077
		20	1.957	41.3	22288

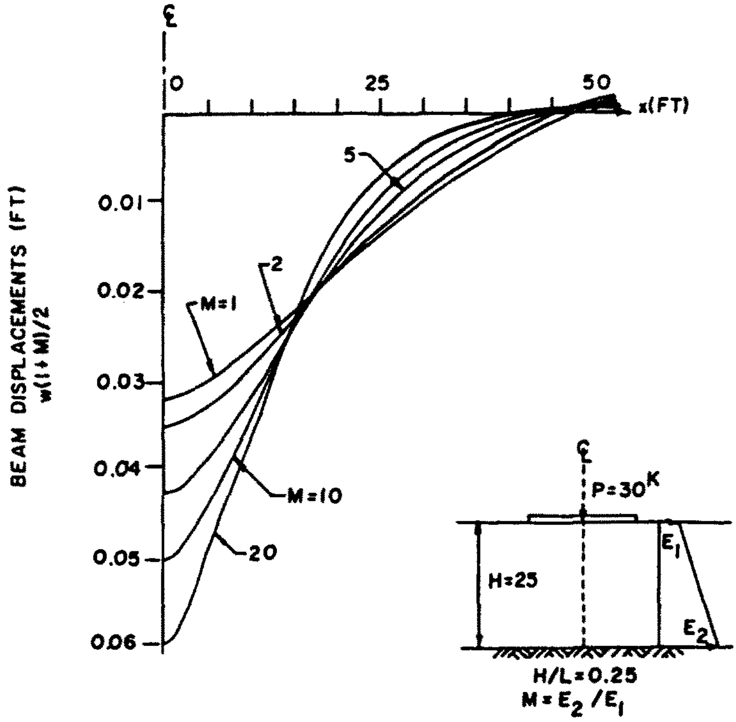

Fig. 5. Beam displacement ($H/L = 0.25$).

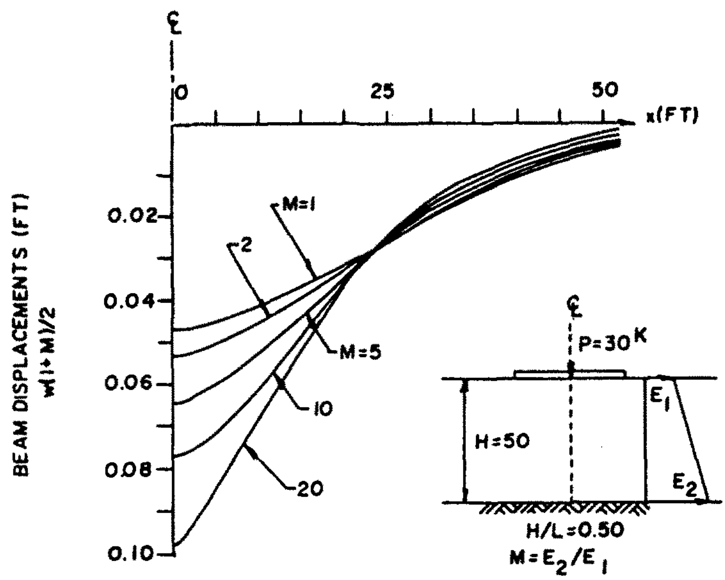

Fig. 6. Beam displacement ($H/L = 0.5$).

settlements. Here, great caution has to be exercised in the interpretation of the results. In general, evaluations of the deformation characteristics of a soil medium are indeed very important in geotechnical engineering practice, especially when they can be achieved in a design office using a personal computer.

Since the method is iterative, one may be concerned about the convergence of the γ parameter. It is observed that convergence to a margin value of γ is achieved for less than seven iterations for uniformly distributed loads and less than 12 iterations for concentrated

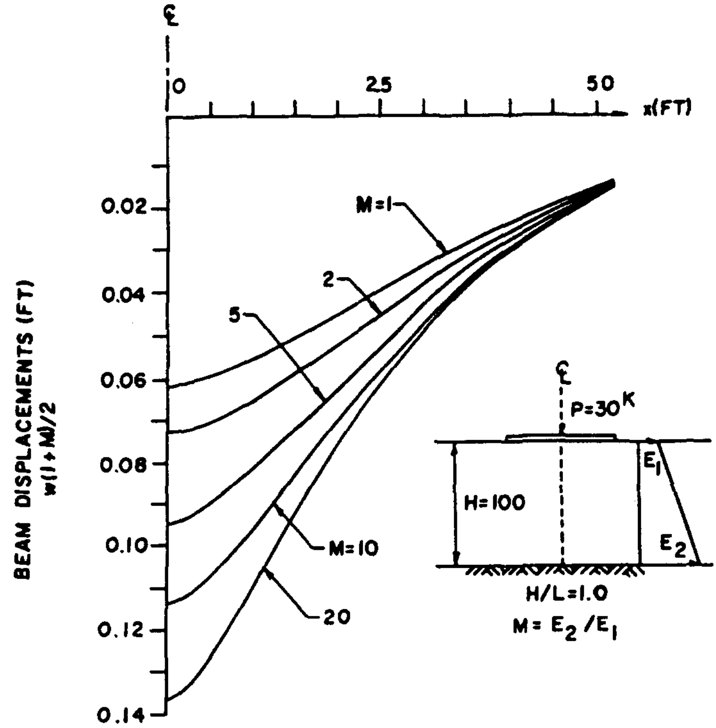

Fig. 7. Beam displacement ($H/L = 1.0$).

loads. A starting value equal to 1.0 is used here. In the author's opinion, this model clarifies a lot of inconsistencies prevailing in the use of modulus of subgrade reaction k in the widely used Winkler model. The method presented here can be expanded to consider layered soils, variations in depth of soil, etc.

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